

# *Lineare Algebra*

## *KW 44*



Nina Gassner

y'all I read the feedback T^T

THANK YOU (???) OMG <3



this is my fav halloween joke:

Why do programmers confuse Halloween and Christmas?



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Why do programmers confuse Halloween and Christmas?

**Because OCT 31 = DEC 25.**



I found this yesterday:

Ein Besucher hört aus einem Grab eine Stimme, die Polynome in Faktoren zerlegt.  
Er fragt den Friedhofswärter, was da los sei.  
Ach, das ist Euler.



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Er fragt den Friedhofswärter, was da los sei.  
Ach, das ist Euler. **He's decomposing.**



oder super passend:

What happens to a matrix when it dies?



oder super passend:

What happens to a matrix when it dies?

It decomposes.

$$(A = CR)$$



# *Gauss-Jordan-Elimination*

ein (diesmal)  
gaaanz kurzer recap!

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$${}^{m \times n} \left( \begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & \mathbf{B} & \\ & & \end{bmatrix} \right) {}^{m \times m}$$

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$$\begin{array}{ccc} m \times n & \left( \begin{bmatrix} A \end{bmatrix}, \begin{bmatrix} B \end{bmatrix} \right) & m \times m \\ & \downarrow & \\ & \begin{bmatrix} A & | & B \end{bmatrix} & \end{array}$$

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$$\begin{array}{ccc} m \times n & \left( \left[ \begin{array}{c} A \end{array} \right], \left[ \begin{array}{c} B \end{array} \right] \right) & m \times m \\ & \downarrow & \\ & \left[ \begin{array}{c|c} A & B \end{array} \right] & \\ & \downarrow M \cdot, M = M_\ell \cdots M_2 M_1 & \\ & \left[ \begin{array}{c|c} MA & MB \end{array} \right] & \end{array}$$

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 & \downarrow & \\
 & \left( \begin{bmatrix} MA \end{bmatrix}, j_1, j_2, \dots, j_r, \begin{bmatrix} MB \end{bmatrix} \right) &
 \end{array}$$

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 \downarrow \\
 \left( \left[ \begin{array}{c|c} MA & \end{array} \right], j_1, j_2, \dots, j_r, \left[ \begin{array}{c} MB \end{array} \right] \right) \\
 = \left( \left[ \begin{array}{c|c} R & \end{array} \right], j_1, j_2, \dots, j_r, \left[ \begin{array}{c} C \end{array} \right] \right)
 \end{array}$$

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$$\left( \begin{bmatrix} & R & \end{bmatrix}, \begin{matrix} j_1, j_2, \dots, j_r \\ \uparrow \end{matrix}, \begin{bmatrix} & c & \end{bmatrix} \right)$$



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Spalten, wo in  $R e_1, \dots, e_r$  stehen



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$$R = \left[ \begin{array}{c|c} R' & \\ \hline 0 & \end{array} \right] \begin{matrix} r \\ m-r \end{matrix}$$

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$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{Gauss-Jordan}} R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$



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$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{pmatrix} \xrightarrow{\text{Gauss-Jordan}} R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$



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so many calculation-questions!

HS25

a) [3 Points] Consider the linear system  $A\mathbf{x} = \mathbf{b}$  with

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ -2 & -6 & 5 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the unique solution  $\mathbf{x} \in \mathbb{R}^3$  to the linear system.



# *Gauss-Jordan-Elimination*

so many calculation-questions!

FS21

1. **Gauss-Elimination** Gegeben ist die Matrix **A** und der Vektor **b**

/20

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 4 & 8 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 6 \\ -4 \end{pmatrix}.$$

- a) Der Vektor  $\mathbf{x} = (0 \ 2 \ -1 \ 0)^\top$  ist eine partikuläre Lösung der Gleichung  $\mathbf{Ax} = \mathbf{b}$ .  
Bestimmen Sie die allgemeine Lösung dieser Gleichung.

/10



# *Gauss-Jordan-Elimination*

mit allgemeiner Lösungsmengen-Bestimmung  
kann man dann auch viele MC-Fragen lösen :)



# *Gauss-Jordan-Elimination*

let's solve this later if there is time left! xD

Let  $A \in \mathbb{R}^{4 \times 5}$  be the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & 2 \\ 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 2 & 3 & -1 \end{pmatrix}.$$

Applying Gauss-Jordan elimination to  $A$  yields

$$R = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Construct the matrices  $C$  and  $R'$  of the CR decomposition  $A = CR'$ .



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# Vector Space

**Definition 4.1:** A (real) vector space<sup>1</sup> is a set  $V$  (the *vectors*), equipped with and *closed under* two linear operations:

vector addition:  $\mathbf{v} + \mathbf{w} \in V$  for all  $\mathbf{v}, \mathbf{w} \in V$ ,

scalar multiplication:  $\lambda \mathbf{v} \in V$  for all  $\mathbf{v} \in V$  and all  $\lambda \in \mathbb{R}$ ,

satisfying the following *axioms* (rules) for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and all  $\lambda, \mu \in \mathbb{R}$ .

1.  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2.  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. There is a vector  $\mathbf{0}$  such that  $\mathbf{v} + \mathbf{0} = \mathbf{v}$  for all  $\mathbf{v}$
4. There is a vector  $-\mathbf{v}$  such that  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5.  $1 \cdot \mathbf{v} = \mathbf{v}$
6.  $(\lambda \cdot \mu) \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$
7.  $\lambda(\mathbf{v} + \mathbf{w}) = \lambda \mathbf{v} + \lambda \mathbf{w}$
8.  $(\lambda + \mu) \mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v}$



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$$(-1)u = -u$$

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# *Vector Space*

Einige Beispiele aus dem Unterricht...

- $\mathbb{R}^m$
- $\mathbb{R}^{\mathbb{R}}$
- $\mathbb{R}[x]$
- $\mathbb{R}^{m \times n}$

(jeweils bezüglich der bestimmten Definitionen der Vektoraddition bzw. Multiplikation mit Skalaren)



# *Subspaces*

**Definition 4.11:** Let  $V$  be a vector space. A *subspace* of  $V$  is a subset  $U \subseteq V$  that is itself a vector space with the same linear operations as  $V$ .

“eine Teilmenge eines Vektorraums, der den Span von sich selber enthält”



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**Lemma 4.12:** Let  $V$  be a vector space and  $U \subseteq V$ . The following two statements are equivalent.

- (i)  $U$  is a subspace of  $V$ .
- (ii)  $U$  is nonempty, and for all  $\mathbf{v}, \mathbf{w} \in U$  and  $\lambda \in \mathbb{R}$ , we have
  - (a)  $\mathbf{v} + \mathbf{w} \in U$  ( $U$  is closed under vector addition),
  - (b)  $\lambda \mathbf{v} \in U$  ( $U$  is closed under scalar multiplication).



# *Subspaces*

für jede Matrix  $A \in \mathbb{R}^{m \times n}$  sind  
 $C(A)$ ,  $R(A)$  und  $N(A)$  Unterräume von  $\mathbb{R}^m$  bzw.  $\mathbb{R}^n$



# *Subspaces*

$C(A)$  kann leer sein.



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$N(A)$  kann genau zwei Elemente beinhalten.



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Die Lösungsmenge von  $Ax=b$  ist immer ein Unterraum.



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# *Subspaces*

Hyperplanes sind immer Unterräume.

**Definition 1.16** (Hyperplane through the origin). *Let  $\mathbf{d} \in \mathbb{R}^m, \mathbf{d} \neq \mathbf{0}$ . The set*

$$H_{\mathbf{d}} = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$$

*is called a hyperplane through the origin.*



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# Subspaces

- a) In this exercise we consider the vector space  $V$  of all real-valued functions on the interval  $[0, 1]$ . In other words, every element  $\mathbf{f} \in V$  is a function  $\mathbf{f} : [0, 1] \rightarrow \mathbb{R}$  and conversely, every function  $\mathbf{f} : [0, 1] \rightarrow \mathbb{R}$  is in  $V$ . Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition  $\mathbf{f} + \mathbf{g}$  of such functions  $\mathbf{f} \in V$  and  $\mathbf{g} \in V$ , and a valid scalar multiplication  $c\mathbf{f}$  for a scalar  $c \in \mathbb{R}$  and  $\mathbf{f} \in V$  defined as follows:

$$\begin{aligned}(\mathbf{f} + \mathbf{g})(x) &:= \mathbf{f}(x) + \mathbf{g}(x) && \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in [0, 1] \\ (c\mathbf{f})(x) &:= c\mathbf{f}(x) && \text{for all } \mathbf{f} \in V, x \in [0, 1] \text{ and } c \in \mathbb{R}.\end{aligned}$$

Prove that

$$U = \{\mathbf{f} \in V : \mathbf{f}(x) = \mathbf{f}(1 - x) \text{ for all } x \in [0, 1]\} \subseteq V$$

is a subspace of  $V$ .



# *Subspaces*

teeny tiny visual help that I used back then :D

→ [linalg.guru](https://linalg.guru)



# *Subspaces*

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Let  $v_1, \dots, v_k$  be linearly independent vectors in  $V$ .  
Are the following statements true?



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A) Every basis of  $V$  must contain all of  $\{v_1, \dots, v_k\}$ .



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- A) Every basis of  $V$  must contain all of  $\{v_1, \dots, v_k\}$ .
- B) If  $k = n$ , then  $\{v_1, \dots, v_k\}$  is a basis of  $V$ .



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Let  $V$  be a vector space with basis  $\{b_1, \dots, b_n\}$ .  
Let  $v_1, \dots, v_k$  be linearly independent vectors in  $V$ .  
Are the following statements true?

A) Every basis of  $V$  must contain all of  $\{v_1, \dots, v_k\}$ .

B) If  $k = n$ , then  $\{v_1, \dots, v_k\}$  is a basis of  $V$ .

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# *Subspaces*

**Definition 4.21:** Let  $V$  be a vector space. A family  $B = (\mathbf{v}_i)_{i \in I}$  of vectors in  $V$  is called a *basis* of  $V$  if (i)  $B$  is linearly independent and (ii)  $B$  spans  $V$ .

- b) Let  $m \in \mathbb{N}^+$ . Consider the set  $\mathcal{D}_m$  of diagonal  $m \times m$  matrices, which is a subspace of  $\mathbb{R}^{m \times m}$ . What is the dimension of  $\mathcal{D}_m$ ? Justify your answer with a proof.



# *Subspaces*

ready for a small kahoot? :D



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*bis nächste Woche!*



<https://linalg.guru>



# Zusatzslides!

b) Write down the inverse  $A^{-1}$  of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}.$$

/1

.7

*Remark:* Recall that the inverse can be found by solving the linear system  $AX = I$  for  $X \in \mathbb{R}^{4 \times 4}$ . Alternatively, you can also approach this as four linear systems  $Ax_1 = e_1$ ,  $Ax_2 = e_2$ ,  $Ax_3 = e_3$ , and  $Ax_4 = e_4$ , where  $x_1, \dots, x_4$  are the columns of  $X$  and  $e_1, \dots, e_4$  are the standard unit vectors (columns of  $I$ ).



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# *Zusatzslides!*

MC2. [2 Points] Let  $A \in \mathbb{R}^{2 \times 2}$  be a matrix satisfying

$$A \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{for any } x, y \in \mathbb{R}.$$

Which of the following statements must be **true**?

- ☐  $A$  is symmetric.
- ☐  $A = 0$ .
- ☐  $\text{rank}(A) < 2$ .



# *Zusatzslides!*

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