Woche 4

Plan

- Hand-in besprechen
- Recap
- Head-TA Lemmas
- In-class & Aufgaben
- Kahoot

Hand-in

2. Nullspace as a hyperplane (hand-in) (★★☆)

Let $\mathbf{v} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ not all zero. Consider the $m \times n$ matrix A of the form

$$A = \begin{bmatrix} | & | & & | \\ \lambda_1 \mathbf{v} & \lambda_2 \mathbf{v} & \cdots & \lambda_n \mathbf{v} \\ | & | & & | \end{bmatrix}$$

a) What is the rank of the matrix A?

b) Prove that the nullspace $\mathbf{N}(A)$ is a hyperplane through the origin.

Recap

Find the CR decomposition for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Find the CR decomposition for

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

In-Class

1. Matrix multiplication with vectors and covectors (in-class) (★☆☆)

Let $n \in \mathbb{N}$. Consider $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ given by $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$.

a) Compute $\mathbf{v}^\top \mathbf{w}$ for all $n \in \mathbb{N}$.

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c) Compute $\mathbf{w}^\top (\mathbf{v}\mathbf{w}^\top)\mathbf{v}$ for all $n \in \mathbb{N}$.

2. Exercise 2.45 (in-class) (★☆☆)

Let $A \in \mathbb{R}^{m \times n}$ of rank r and $C \in \mathbb{R}^{m \times r}$ and $R' \in \mathbb{R}^{r \times n}$ be the matrices in the CR-decomposition of A as given in Theorem 2.44.

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2. Exercise 2.45 (in-class) (★☆☆)

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a) Suppose $r = n$. What are the matrices C and R'?

b) Suppose $r = 0$. What are the matrices C and R'?

Head-TA Lemmas

Let $A \in \mathbb{R}^{n \times n}$ be invertible. Prove Lemma 2.59: the matrix $A^\top$ is invertible, and

$$\left(A^\top\right)^{-1} = \left(A^{-1}\right)^\top.$$

Definition 2.56: Let A be an $m \times m$ matrix. A is invertible if and only if there exists an $m \times m$ matrix B such that $BA = I$ (or $AB = I$, or $AB = BA = I$). In this case, the matrix B is unique and called the *inverse* of A. We denote it by A^{-1}.

Lemma 2.15 (Rank-1 matrices). *Let A be an $m \times n$ matrix. The following two statements are equivalent.*

(i) $\mathbf{rank}(A) = 1$.

(ii) There are nonzero vectors $\mathbf{v} \in \mathbb{R}^m, \mathbf{w} \in \mathbb{R}^n$ *such that* $A = [v_i w_j]_{i=1,j=1}^{m \quad n}$.

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c) Note that this exercise is intended to be more **challenging** than the others! Let $n \in \mathbb{N}^+$ be arbitrary and consider the matrix $A \in \mathbb{R}^{n \times n}$ defined as

$$A := I + B$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix and $B \in \mathbb{R}^{n \times n}$ satisfies $B_{ij} = 1$ for all $i, j \in \{1, 2, \ldots, n\}$ (i.e. all entries of B are 1). Prove that A is invertible.

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c) Let $n \in \mathbb{N}^+$ and let $A \in \mathbb{R}^{n \times n}$ be a matrix satisfying $A^\top + A = 2I$. Prove that $\mathbf{x}^\top A\mathbf{x} = \| x \|^2$ for all $\mathbf{x} \in \mathbb{R}^n$.

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Weitere Aufgaben

b) Let A and B be $m \times m$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

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We say that a square matrix A is *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The minimal $k \in \mathbb{N}$ such that $A^k = 0$ is called the *nilpotent degree* of A.

c) Let A be a nilpotent matrix of degree $k \in \mathbb{N}$, and B be a matrix commuting with A. In particular, note that both A and B are square matrices. Is AB nilpotent? If yes, what can we say about the nilpotent degree of AB?

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d) Let A be an $m \times m$ nilpotent matrix of degree $k \in \mathbb{N}$. Prove that $(I-A)(I+A+\ldots+A^{k-1}) = I$.

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e) Let T be an $m \times m$ upper triangular matrix. Assume that the diagonal of T consists of 0's only. Prove that $T^m = 0$, i.e. T is nilpotent of degree less or equal to m.

Let A and B be $n \times n$ matrices satisfying $A + B = I$. Prove that A and B commute.