

Lineare Algebra

KW 4 1



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Willkommen zurück! :D

Bemerkung zum Quiz

Problem 1. [0.5 Points] Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Write down a vector $\mathbf{x} \in \mathbf{N}(A)$ with $\mathbf{x} \neq \mathbf{0}$.



Willkommen zurück! :D

Bemerkung zum Quiz

Problem 2. [0.5 Points] Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix},$$

with

$$a_{ij} = i - j \quad \text{for all } i, j \in \{1, 2, \dots, m\}.$$

Determine the rank of A_3 . Briefly justify your answer.



Matrix-Matrix-Multiplikation

Lemma 2.15 (Rank-1 matrices). *Let A be an $m \times n$ matrix. The following two statements are equivalent.*

(i) $\text{rank}(A) = 1$.

(ii) *There are nonzero vectors $\mathbf{v} \in \mathbb{R}^m, \mathbf{w} \in \mathbb{R}^n$ such that*

$$A = [v_i w_j]_{i=1, j=1}^m \quad n.$$



Matrix-Matrix-Multiplikation

Rückblick auf Hand-in..:

Der Beweis ist jetzt super einfach für euch! :D

2. Nullspace as a hyperplane (hand-in) (★★☆)

Let $\mathbf{v} \in \mathbb{R}^m \setminus \{0\}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ not all zero. Consider the $m \times n$ matrix A of the form

$$A = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1 \mathbf{v} & \lambda_2 \mathbf{v} & \cdots & \lambda_n \mathbf{v} \\ | & | & \cdots & | \end{bmatrix}$$

a) What is the rank of the matrix A ?



Matrix-Matrix-Multiplikation

Ihr könnt euch auf die Anwendung in späteren Aufgabenblättern freuen!! :)



in-class

1. Matrix multiplication with vectors and covectors (in-class) (★☆☆)

Let $n \in \mathbb{N}$. Consider $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ given by $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$.

a) Compute $\mathbf{v}^\top \mathbf{w}$ for all $n \in \mathbb{N}$.



in-class

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b) Compute $\mathbf{vw}^\top$ when $n = 4$.



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b) Compute $\mathbf{vw}^\top$ when $n = 4$.

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 \\ 6 & 6 & 6 & 6 \\ 8 & 8 & 8 & 8 \end{bmatrix}$$



in-class

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Let $n \in \mathbb{N}$. Consider $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ given by $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$.

c) Compute $\mathbf{w}^\top (\mathbf{v} \mathbf{w}^\top) \mathbf{v}$ for all $n \in \mathbb{N}$.



in-class

... dazu ähnlich: HS24, 3b)

- b) Let $m \in \mathbb{N}^+$ be arbitrary. Denote by $\mathbf{e}_1, \dots, \mathbf{e}_m \in \mathbb{R}^m$ the standard unit vectors, and define $E^{(ij)} := \mathbf{e}_i \mathbf{e}_j^\top \in \mathbb{R}^{m \times m}$ for all $i, j \in \{1, \dots, m\}$. Let now $\ell, k, i, j \in \{1, \dots, m\}$ be arbitrary. Find a simple condition on the indices i, j, ℓ, k that holds if and only if $E^{(\ell k)} E^{(ij)} = 0$. Prove your answer.



CR-decomposition

zuerst einmal ein recap! :D

-> linalg.guru



CR-decomposition

Aufgabe!

Bestimme die CR-Zerlegung $A = CR'$ und den Rang von A .

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$



in-class

2. Exercise 2.45 (in-class) (★☆☆)

Let $A \in \mathbb{R}^{m \times n}$ of rank r and $C \in \mathbb{R}^{m \times r}$ and $R' \in \mathbb{R}^{r \times n}$ be the matrices in the CR-decomposition of A as given in Theorem 2.44.

a) Suppose $r = n$. What are the matrices C and R' ?



in-class

2. Exercise 2.45 (in-class) (★☆☆)

Let $A \in \mathbb{R}^{m \times n}$ of rank r and $C \in \mathbb{R}^{m \times r}$ and $R' \in \mathbb{R}^{r \times n}$ be the matrices in the CR-decomposition of A as given in Theorem 2.44.

b) Suppose $r = 0$. What are the matrices C and R' ?



in-class

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Let $A \in \mathbb{R}^{m \times n}$ of rank r and $C \in \mathbb{R}^{m \times r}$ and $R' \in \mathbb{R}^{r \times n}$ be the matrices in the CR-decomposition of A as given in Theorem 2.44.

b) Suppose $r = 0$. What are the matrices C and R' ?

$$C \in \mathbb{R}^{m \times 0}$$

$$R' \in \mathbb{R}^{0 \times n}$$



CR-decomposition

vlt. können wir das nachher zusammen lösen :D

Zeige für beliebige $x, y \in \mathbb{R}^n$:

$$Ax = Ay \iff R'x = R'y.$$



inverses of matrices

ein Blick auf die Karte...



inverses of matrices

Find the mistake in my proof for:

$$\text{Sei } P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Beweis für: $P = I$

$$\begin{aligned} I = P &\Leftrightarrow PI = P^2 \\ &\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\Leftrightarrow \text{True.} \end{aligned}$$



inverses of matrices

Let's proceed with caution...



inverses of matrices

Let's prove this :D

Lemma 2.59. *Let A and B be invertible $m \times m$ matrices. Then AB is also invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$



inverses of matrices

Let's prove this :D

Lemma 2.60. *Let A be an invertible $m \times m$ matrix. Then the transpose $A^\top$ is also invertible, and*

$$(A^\top)^{-1} = (A^{-1})^\top.$$



inverses of matrices

Let's (maybe) prove this :D

Lemma 2.54. *Let A, B be $m \times m$ matrices such that $BA = I$. Then also $AB = I$.*

right inverse = left inverse? :0



inverses of matrices

HS25

- b) [4 Points] Let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ be linearly independent and $A \in \mathbb{R}^{n \times n}$. Prove that A is invertible if and only if $A\mathbf{v}_1, \dots, A\mathbf{v}_n$ are linearly independent.



inverses of matrices

...umgeschrieben (gleiche Aufgabe!)

gegeben: $v_1, \dots, v_n \in \mathbb{R}^n$ linear unabhängig, $A \in \mathbb{R}^{n \times n}$
zeige:

A invertierbar $\iff Av_1, \dots, Av_n$ linear unabhängig



inverses of matrices

...umgeschrieben (gleiche Aufgabe!)

gegeben: $v_1, \dots, v_n \in \mathbb{R}^n$ linear unabhängig, $A \in \mathbb{R}^{n \times n}$
zeige:

A invertierbar $\Rightarrow Av_1, \dots, Av_n$ linear unabhängig



inverses of matrices

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gegeben: $v_1, \dots, v_n \in \mathbb{R}^n$ linear unabhängig, $A \in \mathbb{R}^{n \times n}$
zeige:

A invertierbar $\Leftrightarrow Av_1, \dots, Av_n$ linear unabhängig



inverses of matrices

It's Kahoot time !!!
:D



bis nächste Woche!



<https://linalg.guru>

