

Woche 3

Plan

- Hand-in besprechen
- Recap
- In-class
- Quiz
- Aufgaben

Hand-in

2. Nullspace as a hyperplane (hand-in) (★★☆)

Let $\mathbf{v} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ not all zero. Consider the $m \times n$ matrix A of the form

$$A = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1 \mathbf{v} & \lambda_2 \mathbf{v} & \cdots & \lambda_n \mathbf{v} \\ | & | & \cdots & | \end{bmatrix}$$

- a) What is the rank of the matrix A ?
- b) Prove that the nullspace $\mathbf{N}(A)$ is a hyperplane through the origin.

In-class

a) Let $n \in \mathbb{N}^+$. Consider the function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$T(\mathbf{x}) = \sum_{k=1}^n kx_k$$

for all $\mathbf{x} = (x_1 \ x_2 \ \cdots \ x_n)^\top \in \mathbb{R}^n$. Prove that T is a linear functional.

b) Let $n \in \mathbb{N}^+$ with $n \geq 2$ be arbitrary. Consider the function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$T(\mathbf{x}) = \sum_{k=1}^n (x_k)^k$$

for all $\mathbf{x} = (x_1 \ x_2 \ \cdots \ x_n)^\top \in \mathbb{R}^n$. Is T a linear functional?

Alte Klausuraufgaben

3. Let $n \in \mathbb{N}^+$ be arbitrary and fix $\mathbf{v} \in \mathbb{R}^n$. Consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as $f(\mathbf{x}) := \|\mathbf{x} - \mathbf{v}\|^2 - \mathbf{x}^\top \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Which of the following statements must be **true**? *Note: only one answer is correct.*

- (a) f is a linear transformation if and only if $\mathbf{v} \neq \mathbf{0}$.
- (b) f is a linear transformation if and only if $\mathbf{v} = \mathbf{0}$.
- (c) f is a linear transformation if and only if $\|\mathbf{v}\| = 1$.
- (d) It is not possible to choose $\mathbf{v}$ such that f is a linear transformation.

a) Let $n \in \mathbb{N}^+$ and assume that $A \in \mathbb{R}^{n \times n}$ satisfies $AA^\top = I$ and $A^2 = I$. Prove that A is symmetric.

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c) Let $n \in \mathbb{N}^+$ and let $A \in \mathbb{R}^{n \times n}$ be a matrix satisfying $A^\top + A = 2I$. Prove that $\mathbf{x}^\top A \mathbf{x} = \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^n$.

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Weitere Aufgaben

b) Let A and B be $m \times m$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

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We say that a square matrix A is *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The minimal $k \in \mathbb{N}$ such that $A^k = 0$ is called the *nilpotent degree* of A .

- c)** Let A be a nilpotent matrix of degree $k \in \mathbb{N}$, and B be a matrix commuting with A . In particular, note that both A and B are square matrices. Is AB nilpotent? If yes, what can we say about the nilpotent degree of AB ?

- b)** Let A and B be $m \times m$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

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- d)** Let A be an $m \times m$ nilpotent matrix of degree $k \in \mathbb{N}$. Prove that $(I - A)(I + A + \dots + A^{k-1}) = I$.

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- e)** Let T be an $m \times m$ upper triangular matrix. Assume that the diagonal of T consists of 0's only. Prove that $T^m = 0$, i.e. T is nilpotent of degree less or equal to m .

Let A and B be $n \times n$ matrices satisfying $A + B = I$. Prove that A and B commute.