

Woche 2

Plan

- Hand-in kurz besprechen
- Recap
- *Pause (Quiz abholen)*
- Aufgaben
- Kahoot

Aufgaben

Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$f \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) := \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \quad \text{for all } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3.$$

Write down the unique matrix $P \in \mathbb{R}^{3 \times 3}$ that satisfies $Px = f(x)$ for all $x \in \mathbb{R}^3$.

c) Let $n \in \mathbb{N}^+$. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} x_1 - x_2 \\ 3x_2 \\ 2x_1 + 2x_2 \end{bmatrix}$$

for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. Write down the unique matrix A that satisfies $A\mathbf{x} = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$.

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Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$.

a) Calculate A_m for $m \in \{2, 3, 4\}$.

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with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$.

- a)** Calculate A_m for $m \in \{2, 3, 4\}$.
- b)** Determine the rank of A_m . You need to motivate your answer.

Consider the 3×3 matrix

$$A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} = \mathbf{v}\mathbf{w}^\top$$

with $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$. Recall that by Lemma 2.21, A has rank 1 if and only if both $\mathbf{v}$ and $\mathbf{w}$ are non-zero. Otherwise, A has rank 0.

- a)** Assume now $v_1 \neq 0$ and $w_1 \neq 0$. Find a non-zero vector $\mathbf{x}$ (expressed in terms of $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$) that satisfies $A\mathbf{x} = \mathbf{0}$ (non-zero means that it cannot be the zero-vector $\mathbf{0}$).

Consider the 3×3 matrix

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- a) Assume now $v_1 \neq 0$ and $w_1 \neq 0$. Find a non-zero vector $\mathbf{x}$ (expressed in terms of $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$) that satisfies $A\mathbf{x} = \mathbf{0}$ (non-zero means that it cannot be the zero-vector $\mathbf{0}$).
- b) Recall that we call a set of vectors $H \subseteq \mathbb{R}^m$ a hyperplane of $\mathbb{R}^m$ if and only if there exists a non-zero vector $\mathbf{d} \in \mathbb{R}^m$ such that $H = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$. We still assume that $v_1 \neq 0$ and $w_1 \neq 0$. Consider the set of vectors $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. Prove that $\mathcal{L}$ is a hyperplane of $\mathbb{R}^3$.

c) [**3 Points**] Consider the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad , \quad \mathbf{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} .$$

Determine a matrix $A \in \mathbb{R}^{2 \times 2}$ satisfying

$$A(\mathbf{u} + \mathbf{v}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and}$$

$$A(\mathbf{u} - \mathbf{v}) = \begin{pmatrix} 1 \\ 5 \end{pmatrix} .$$

a) **[3 Points]** Let $n \in \mathbb{N}^+$ be arbitrary and consider two arbitrary vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ satisfying $\mathbf{v}^\top \mathbf{w} \leq 0$. Prove that

$$\|\mathbf{v} - \mathbf{w}\|^2 \geq \max\{\|\mathbf{v}\|^2, \|\mathbf{w}\|^2\}.$$

Moreover, show that equality holds if and only if $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$.