

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \end{bmatrix}$$

Feedback Serien

- Gute Arbeit von allen!
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In-Class

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$.

a) Calculate A_m for $m \in \{2, 3, 4\}$.

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$.

b) Determine the rank of A_m . You need to motivate your answer.

Extra (Exam)

MC1. [2 Points] Let $A \in \mathbb{R}^{4 \times 5}$ with $\text{rank}(A) = 4$. Consider the linear system $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^4$. Which of the following statements must be **true**?

- ☐ The system has always no solution.
- ☐ The system has always a unique solution.
- ☐ The system has always multiple solutions.
- ☐ Sometimes, the system has no solution.

HS25

Discussion Proof Exchange Lemma

Lemma 1.26 (The span of m linearly independent vectors is $\mathbb{R}^m$). *Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^m$ be linearly independent. Then $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) = \mathbb{R}^m$.*

Extra Exercises

Does there exist a **linear** map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

$$T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad T \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ?$$

Justify your answer.

What general fact can we derive
from this?

Is the converse of (c) true?

That is: if a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies

$$f(0) = 0,$$

must f be linear?

Give a proof or a counterexample.

Extra Questions

Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$f \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) := \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \quad \text{for all } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3.$$

Write down the unique matrix $P \in \mathbb{R}^{3 \times 3}$ that satisfies $Px = f(x)$ for all $x \in \mathbb{R}^3$.

Consider the 3×3 matrix

$$A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} = \mathbf{v}\mathbf{w}^\top$$

with $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$. Recall that by Lemma 2.21, A has rank 1 if and only if both $\mathbf{v}$ and $\mathbf{w}$ are non-zero. Otherwise, A has rank 0.

- a)** Assume now $v_1 \neq 0$ and $w_1 \neq 0$. Find a non-zero vector $\mathbf{x}$ (expressed in terms of $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$) that satisfies $A\mathbf{x} = \mathbf{0}$ (non-zero means that it cannot be the zero-vector $\mathbf{0}$).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity I

$$a_{ij} = \delta_{ij}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

diagonal

$$i \neq j : a_{ij} = 0$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

upper triangular

$$i > j : a_{ij} = 0$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 7 & 5 \end{bmatrix}$$

lower triangular

$$i < j : a_{ij} = 0$$

Gedankenanstoss

Gilt $\text{rank}(A) = \text{rank}(A^T)$ immer?