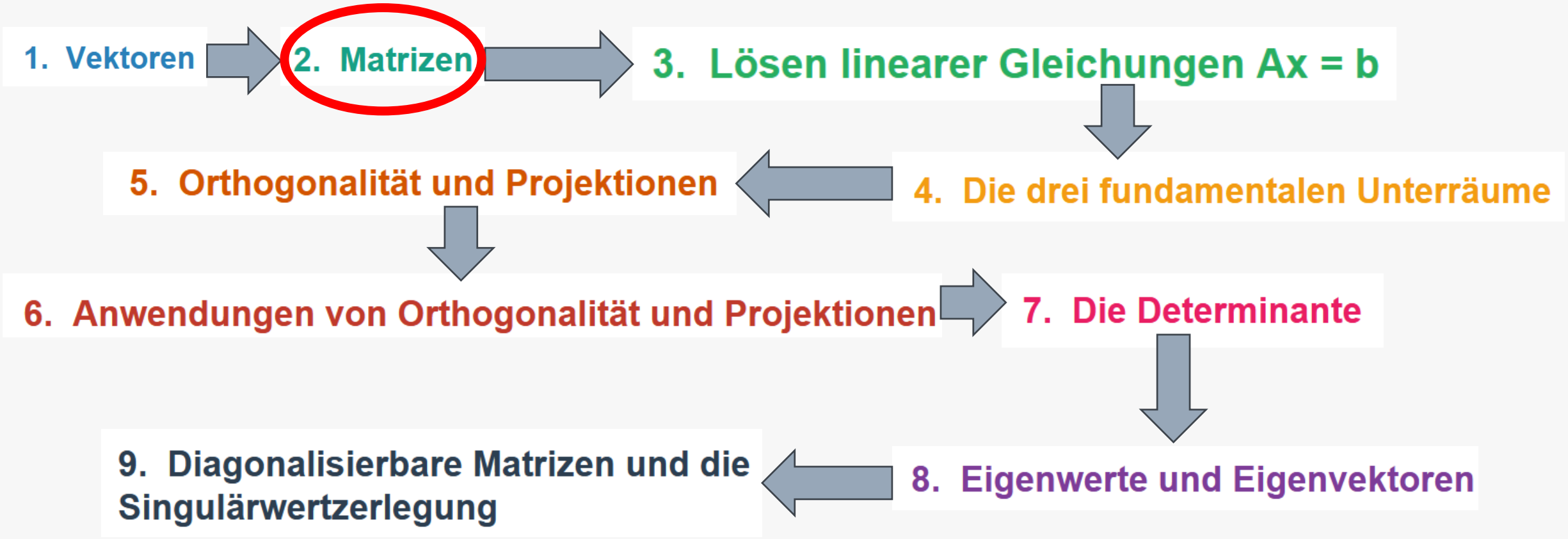
LINEAR ALGEBRA

WEEK 2

Roadmap


1. Vektoren
2. Matrizen
3. Lösen linearer Gleichungen Ax = b
4. Die drei fundamentalen Unterräume
5. Orthogonalität und Projektionen
6. Anwendungen von Orthogonalität und Projektionen
7. Die Determinante
8. Eigenwerte und Eigenvektoren
9. Diagonalisierbare Matrizen und die Singulärwertzerlegung

FEEDBACK

a) Let $\mathbf{0} \in \mathbb{R}^m$ denote the vector whose entries are all zero. We say that a set L is a line through the origin in $\mathbb{R}^m$ if and only if there exists $\mathbf{w} \in \mathbb{R}^m$ with $\mathbf{w} \neq \mathbf{0}$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. Let now L be a line through the origin in $\mathbb{R}^m$ and let $\mathbf{u}$ be an arbitrary non-zero element of L. Prove that $L = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$.

$\exists u \in \mathbb{R}^m : L = \{\lambda u : \lambda \in \mathbb{R}\}$ ①

$\Rightarrow u = \lambda_u w$ ②

z.z. $L = \{\lambda u : \lambda \in \mathbb{R}\}$

"$\subseteq$": betrachte $v \in L : v = \lambda_v w$

$\Rightarrow v = \frac{\lambda_v}{\lambda_u} u$ ②

$\Rightarrow v \in \{\lambda u : \lambda \in \mathbb{R}\}$

"$\supseteq$": betrachte $v' \in \{\lambda u : \lambda \in \mathbb{R}\} : v' = \lambda_{v'} u$

$\Rightarrow v' = \lambda_{v'} u = \lambda_{v'} \lambda_u w$

$\Rightarrow v' \in L$

b) For two lines through the origin L_1 and L_2 in $\mathbb{R}^m$, prove that we have either $L_1 \cap L_2 = \{0\}$ or $L_1 \cap L_2 = L_1 = L_2$.

$L_1 = \{\lambda w_1 : \lambda \in \mathbb{R}\}$ ①
$L_2 = \{\lambda w_2 : \lambda \in \mathbb{R}\}$ ② $\quad w_1, w_2 \in \mathbb{R}^m$

z.z.: $L_1 \cap L_2 \overset{①}{=} \{0\}$ oder $L_1 \cap L_2 \overset{②}{=} L_1 = L_2$

1) Beide Fälle sind unmöglich, weil $\{0\} \neq L_1$ oder $\{0\} \neq L_2$

2) $0 = 0 \cdot w_1$ und $0 = 0 \cdot w_2 \Rightarrow 0 \in L_1$ und $0 \in L_2$
$\Rightarrow 0 \in L_1 \cap L_2$

3) Wir haben 2 Fälle:

Fall 1 ($L_1 \cap L_2 = \{0\}$):
dies impliziert automatisch ①

Fall 2 ($L_1 \cap L_2 \neq \{0\}$):
$L_1 \cap L_2 \neq \emptyset \Rightarrow \exists u \in \mathbb{R}^m : u \in L_1 \cap L_2$ ($u \neq 0$, weil $L_1 \cap L_2 \neq \{0\}$)

$$\left.\begin{array}{l} u \in L_1 \overset{a)}{\Rightarrow} L_1 = \{\lambda u : \lambda \in \mathbb{R}\} \\ u \in L_2 \overset{a)}{\Rightarrow} L_2 = \{\lambda u : \lambda \in \mathbb{R}\} \end{array}\right\} \Rightarrow L_1 = L_2 = L_1 \cap L_2 \ ②$$

Da wir alle Fälle abgedeckt haben und entweder ① oder ② gilt, gilt die Aussage als bewiesen.

c) Consider an arbitrary line through the origin L in $\mathbb{R}^2$. Prove that L is a hyperplane, i.e. find a vector $\mathbf{d} \neq 0$ such that

$$L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}.$$

$w \neq 0$ ③

↳ per Def.: $\exists w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} : L = \{\lambda w : \lambda \in \mathbb{R}\}$ ①

↳ wir suchen $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, sodass $w \cdot d = 0$

$\Rightarrow d_1 := -w_2$, $d_2 := w_1$ ②

z.z.: $L = \{v \in \mathbb{R}^2 : v \cdot d = 0\}$

"$\subseteq$": wähle $u \in L$: $u = \lambda w$ ①

$u \cdot d = (\lambda w) \cdot d = \lambda (w \cdot d) \overset{②}{=} \lambda \cdot 0 = 0$

$\Rightarrow u \in \{v \in \mathbb{R}^2 : v \cdot d = 0\}$

"$\supseteq$": wähle $u \in \{v \in \mathbb{R}^2 : v \cdot d = 0\}$

$u \cdot d = u_1 d_1 + u_2 d_2 = 0$

$\overset{②}{\Rightarrow} u \cdot d = -u_1 w_2 + u_2 w_1 = 0$

$\Rightarrow u_2 w_1 = u_1 w_2$

$\Rightarrow u_2 = \frac{w_2}{w_1} u_1 = \frac{u_1}{w_1} w_2$

für $w_1 \neq 0$:

wähle $\lambda = \frac{u_1}{w_1} \Rightarrow u_1 = \lambda w_1$ und $u_2 = \lambda w_2$

$\Rightarrow u = \lambda w$

$\Rightarrow u \in L$ ①

für $w_2 \neq 0$: Beweis analog

↳ wir müssen entweder $w_1 \neq 0$ oder $w_2 \neq 0$ Fälle prüfen wegen $w \neq 0$ ③

↳ Proof by case distinction (mehr im DiskMath Skript in Kapitel "Proof Patterns")

FRAGEN?

Definition 2.10 ((In)dependent column and (column) rank of a matrix). *Let*

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

be an $m \times n$ matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$. Column $\mathbf{v}_j$ is called independent *if $\mathbf{v}_j$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$. Otherwise, $\mathbf{v}_j$ is called* dependent. *The* rank *of A, written as* $\mathbf{rank}(A)$, *is the number of independent columns of A. The rank is sometimes also called the* column rank *to distinguish it more explicitly from the* row rank, *see Definition 2.14 below.*

In-class

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$.

a) Calculate A_m for $m \in \{2, 3, 4\}$.

In-class

$$A_2 = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}, \text{ and } A_4 = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}.$$

In-class

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$.

a) Calculate A_m for $m \in \{2, 3, 4\}$.

b) Determine the rank of A_m. You need to motivate your answer.

Aufgaben

a) Sei $A \in \mathbb{R}^{m \times n}$ mit $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

zeige: (1) $0 \in C(A)$ und (2) $\forall j \in [n] : a_j \in C(A)$

Aufgaben

b) Sei $A \in \mathbb{R}^{m \times n}$

zeige:

① $\forall y, z \in C(A) \; \forall \lambda \in \mathbb{R}: \; y + \lambda z \in C(A)$

② $\forall u, v \in N(A) \; \forall \lambda \in \mathbb{R}: \; u + \lambda v \in N(A)$

Aufgaben

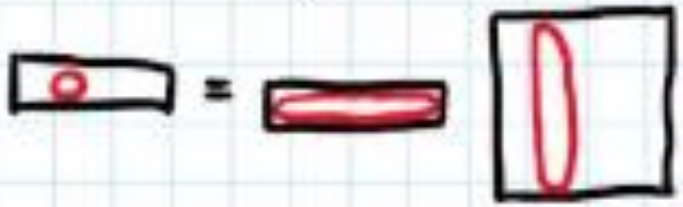
c) Sei $A \in \mathbb{R}^{m \times n}$ und $b \in \mathbb{R}^m$. Es gilt für $x_0 \in \mathbb{R}^n$ $Ax_0 = b$

zeige: $\{x \in \mathbb{R}^n : Ax = b\} = \{x_0 + z : z \in N(A)\}$

Proof: $(Ax)^T = x^T A^T$, $A \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$ (oder $\mathbb{R}^{n \times 1}$) $\Rightarrow Ax \in \mathbb{R}^m$ (oder $\mathbb{R}^{m \times 1}$)

① $(Ax)_{i1} = \sum_{j=1}^{n} a_{ij} x_j \Rightarrow (Ax)^T_{1i} = \sum_{j=1}^{n} a_{ij} x_j$

② $(A^T)_{ji} = a_{ij} \Rightarrow (x^T A^T)_{1i} = \sum_{j=1}^{n} x_j (A^T)_{ji} = \sum_{j=1}^{n} x_j a_{ij}$



$$(Ax)^T_{1i} = \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} x_j a_{ij} = (x^T A^T)_{1i}$$

$$\Rightarrow (Ax)^T = x^T A^T$$

KAHOOT