

Lineare Algebra

KW 39



Nina Gassner

Willkommen zurück! :D

Zeit für das Quiz! :)



Willkommen zurück! :D

kurzer Kommentar zu Abgaben

- Anfang eines Beweises
- Ende eines Beweises



Willkommen zurück! :D

kurzer Kommentar zu Abgaben

- Anfang eines Beweises
- Ende eines Beweises

weiter so! :D



Willkommen zurück! :D

Ihr habt gesehen:

- Skalarprodukt
- euklidische Norm
- Cauchy-Schwarz-Ungleichung, Dreiecksungleichung
- Orthogonalität, Hyperebenen
- lineare Unabhängigkeit
- Span



Skalarprodukt

Visualisierung! :D

$$\begin{aligned} & \mathbf{v}^T \mathbf{w} \\ = & |\mathbf{v}| \cdot \text{signed shadow of } \mathbf{w} \text{ onto } \mathbf{v} \\ = & |\mathbf{w}| \cdot \text{signed shadow of } \mathbf{v} \text{ onto } \mathbf{w} \end{aligned}$$



Skalarprodukt

Definition 1.9 (Scalar product). *Let*

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \in \mathbb{R}^m.$$

The scalar product of $\mathbf{v}$ and $\mathbf{w}$ is the number

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + \cdots + v_m w_m = \sum_{i=1}^m v_i w_i.$$

alternative Notationen: $\langle v, w \rangle$, $v^T w$

Was ist v^T ?



Skalarprodukt

Es ist oft sehr hilfreich, Skalarprodukte mittels Summenformeln umzuschreiben.
(v.a. bei Beweisen)

Aufgabe:

Seien $(v, w \in \mathbb{R}^n)$ mit $(0 \leq v_i \leq w_i)$ für alle i .

Zeige: $v \cdot w \geq \|v\|^2$.



Skalarprodukt

Diese Eigenschaft ist auch sehr nützlich:

$$v^T v = 0 \Leftrightarrow v = 0$$

Aufgabe:

$$u, v \in \mathbb{R}^m, \quad x \in \text{Span}(u, v) \\ x^T v = 0, x^T u = 0$$

Zeige: $x = 0$



euklidische Norm, unit vectors

geometrisch: «Länge»

Definition 1.11 (Euclidean norm). *Let $\mathbf{v} \in \mathbb{R}^m$. The Euclidean norm of $\mathbf{v}$ is the number*

$$\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

$$\left\| \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_m^2} = \sqrt{\sum_{i=1}^m v_i^2}.$$



euklidische Norm, unit vectors

geometrisch: «Länge»

2D: wieso?

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euklische Norm, unit vectors

aus Pruefungen

FS24:

b) Let $n \in \mathbb{N}^+$ and let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

be arbitrary vectors in $\mathbb{R}^n$ with non-negative entries, i.e. $v_i \geq 0$ and $w_i \geq 0$ for all $i \in \{1, 2, \dots, n\}$. Define $u_i := \min\{v_i, w_i\}$ for all $i \in \{1, 2, \dots, n\}$ and consider the vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Prove that $\mathbf{v} \cdot \mathbf{w} \geq \|\mathbf{u}\|^2$.

/3 ·2

HS23:

a) Let $n \in \mathbb{N}^+$ and let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be arbitrary. Prove that

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\right)$$

by direct computation.

/3 ·1



euklische Norm, unit vectors

$\frac{v}{\|v\|}$ ist ein Einheitsvektor!

- geometrisch?
- arithmetisch?



Cauchy-Schwarz-Ungleichung!

1.2.3 Cauchy-Schwarz inequality

As innocent as it looks, the importance of this inequality cannot be overestimated.

Lemma 1.12 (Cauchy-Schwarz inequality). *For any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$,*

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

Moreover, equality holds exactly if one vector is a scalar multiple of the other.

remark: Wir haben untere UND obere Schranke für das Skalarprodukt!



Cauchy-Schwarz-Ungleichung!

wann Gleichheit? ($u \neq 0$)

$$|u^\top v| = \|u\| \|v\|$$
$$\Leftrightarrow$$

$$\|u + v\| = \|u\| + \|v\|$$
$$\Leftrightarrow$$



Cauchy-Schwarz-Ungleichung!

wann Gleichheit? ($u \neq 0$)

$$\begin{aligned} |u^\top v| &= \|u\| \|v\| \\ \Leftrightarrow \exists \lambda \in \mathbb{R}: v &= \lambda u \end{aligned}$$

$$\begin{aligned} \|u + v\| &= \|u\| + \|v\| \\ \Leftrightarrow \end{aligned}$$



Cauchy-Schwarz-Ungleichung!

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$$\begin{aligned} \|u + v\| &= \|u\| + \|v\| \\ \Leftrightarrow \exists \lambda \geq 0: v &= \lambda u \end{aligned}$$



Cauchy-Schwarz-Ungleichung!

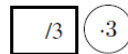
aus einer Pruefung

FS24

- c) Note that this exercise is intended to be more **challenging** than the others! Let $n \in \mathbb{N}^+$. Let $a_1, a_2, \dots, a_n \in \mathbb{R}$ be arbitrary, and let $b_1, b_2, \dots, b_n \in \mathbb{R}$ be arbitrary with $b_i > 0$ for all $i \in \{1, 2, \dots, n\}$. Prove that

$$\sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}.$$

Hint: Use the Cauchy-Schwarz inequality which says that $\|\mathbf{u}\| \|\mathbf{v}\| \geq |\mathbf{u} \cdot \mathbf{v}|$ holds for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.



Cauchy-Schwarz-Ungleichung!

meistens einzige Schwierigkeit bei solchen Aufgaben:

das Finden der «richtigen Vektoren»!

→ lässt sich sehr gut üben :)

SIGNAL: *Ungleichung*

⇒ *Cauchy – Schwarz – /Dreieck – Ungleichung* ausprobieren!



Cauchy-Schwarz-Ungleichung!

Aufgabe

Seien $x, y, z \in \mathbb{R}$ mit $x^2 + y^2 + z^2 = 4$.

Zeige: $2x - y + 2z \leq 6$.



Cauchy-Schwarz-Ungleichung!

Aufgabe

Seien $n \geq 1$ und $x_1, \dots, x_n \in \mathbb{R}$ mit $\sum_{i=1}^n x_i^2 = 1$.

Zeige mit Cauchy-Schwarz: $\left| \sum_{i=1}^n x_i \right| \leq \sqrt{n}$.



Cauchy-Schwarz-Ungleichung!

Aufgabe 3! :D



Cauchy-Schwarz-Ungleichung!

falls Zeit übrig:

- Beweis der Cauchy-Schwarz-Ungleichung
- Beweis der Dreiecksungleichung



Lineare Unabhängigkeit, Span

Definition 1.21 (Linear (in)dependence). Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$ are linearly dependent if at least one of them is a linear combination of the others, i.e. there is an index $k \in [n]$ and scalars λ_j such that

$$\mathbf{v}_k = \sum_{\substack{j=1 \\ j \neq k}}^n \lambda_j \mathbf{v}_j.$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent.

Lemma 1.22 (Alternative definitions of linear dependence). Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$. The following statements are equivalent (meaning that they are either all true, or all false).

- (i) At least one of the vectors is a linear combination of the other ones. (This means, the vectors are linearly dependent according to Definition 1.21.)
- (ii) There are scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ besides $0, 0, \dots, 0$ such that $\sum_{j=1}^n \lambda_j \mathbf{v}_j = \mathbf{0}$. We also say that $\mathbf{0}$ is a nontrivial linear combination of the vectors.
- (iii) At least one of the vectors is a linear combination of the previous ones.



Lineare Unabhängigkeit, Span

Vektoren sind linear unabhängig, falls jeder eine neue «Richtung» hinzufügt, in welche man vorher nicht gehen konnte!

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einzigster Weg zum
Ursprung:
“gar nicht erst loslaufen”
(credit: Jonas)

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Lineare Unabhängigkeit, Span

Definition 1.25 (Span). Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$. Their span is the set of all linear combinations. In formulas,

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) := \left\{ \sum_{j=1}^n \lambda_j \mathbf{v}_j : \lambda_j \in \mathbb{R} \text{ for all } j \in [n] \right\}.$$

Lemma 1.26. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$, and let $\mathbf{v} \in \mathbb{R}^m$ be a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Then

$$\underbrace{\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)}_S = \underbrace{\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v})}_T.$$

Das Hinzufügen linear abhängiger Vektoren ändert den Span nicht!
Span leerer Menge?



Lineare Unabhängigkeit, Span

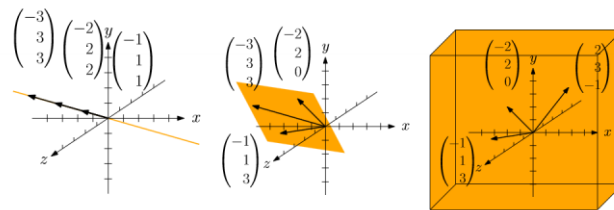
3. Geometry of linear combinations (in-class) (★☆☆)

In this exercise, you are asked to sketch sets of points in $\mathbb{R}^3$. No formal justification is required.

a) Draw the set of linear combinations $\left\{ \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$.

b) Draw the set of linear combinations $\left\{ \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$.

c) Draw the set of linear combinations $\left\{ \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$.



ein neuer Blick auf die Aufgaben letzter Woche!

Was hier gefragt war: Was ist der Span der 3 Vektoren?

bei linearer Abhängigkeit: nicht verändert



Lineare Unabhängigkeit, Span

linear unabhängig?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$



Lineare Unabhängigkeit, Span

linear unabhängig?
intuitiver Beweis?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$



Lineare Unabhängigkeit, Span

Eindeutigkeit der Linearkombination mit unabhängigen Vektoren!

Lemma 1.24. *Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$ be linearly independent, $\mathbf{v} \in \mathbb{R}^m$. Let*

$$\mathbf{v} = \sum_{j=1}^n \lambda_j \mathbf{v}_j = \sum_{j=1}^n \mu_j \mathbf{v}_j$$

be two ways of writing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Then $\lambda_j = \mu_j$ for all $j \in [n]$.

Beweis?



in-Class Aufgabe

1. Lines in $\mathbb{R}^m$ (in-class) (★★☆)

- a) Let $\mathbf{0} \in \mathbb{R}^m$ denote the vector whose entries are all zero. We say that a set L is a line through the origin in $\mathbb{R}^m$ if and only if there exists $\mathbf{w} \in \mathbb{R}^m$ with $\mathbf{w} \neq \mathbf{0}$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. Let now L be a line through the origin in $\mathbb{R}^m$ and let $\mathbf{u}$ be an arbitrary non-zero element of L . Prove that $L = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$.



in-Class Aufgabe

$$\forall L \quad \forall v \in (L \setminus \{0\})$$

$$L = \{ \lambda v \mid \lambda \in \mathbb{R} \}$$

(L)



in-Class Aufgabe

$$\forall L \quad \forall v \in (L \setminus \{0\})$$

$$L \subseteq L'$$

und

$$L' \subseteq L$$



in-Class Aufgabe

- b) For two lines through the origin L_1 and L_2 in $\mathbb{R}^m$, prove that we have either $L_1 \cap L_2 = \{\mathbf{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$. A



in-Class Aufgabe

$$\forall L_1, L_2$$

$$L_1 \cap L_2 = \{0\} \quad (A)$$

xor

$$L_1 \cap L_2 = L_1 = L_2 \quad (B)$$



in-Class Aufgabe

$\forall L_1, L_2$

A oder B

UND

NICHT A und B



in-Class Aufgabe

- c) Consider an arbitrary line through the origin L in $\mathbb{R}^2$. Prove that L is a hyperplane, i.e. find a vector $\mathbf{d} \neq \mathbf{0}$ such that

$$L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}.$$



Kahoot time! :D



bis nächste Woche!



<https://linalg.guru>

